

# SURFACES WITH $K^2 = 2\chi - 2$ AND $p_g \geq 5$

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**ABSTRACT.** This note describes minimal surfaces  $S$  of general type satisfying  $p_g \geq 5$  and  $K^2 = 2p_g$ . For  $p_g \geq 8$  the canonical map of such surfaces is generically finite of degree 2 and the bulk of the paper is a complete characterization of such surfaces with non birational canonical map. It turns out that if  $p_g \geq 13$ ,  $S$  has always an (unique) genus 2 fibration, whose non 2-connected fibres can be characterized, whilst for  $p_g \leq 12$  there are two other classes of such surfaces with non birational canonical map.

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 Canonical map  
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## 1. INTRODUCTION

Noether's well known inequality states that a minimal surface of general type satisfies  $K^2 \geq 2\chi - 6$ . Surfaces with  $K^2 < 2\chi$  are always regular and Horikawa completely classified minimal surfaces satisfying  $K^2 = 2\chi - 6, 2\chi - 5$  and  $2\chi - 4$  ([16], [17], [18], [19]). Some aspects of surfaces with  $K^2 = 2\chi - 3$  have been studied by other authors (e.g. [29]).

In this paper we characterize minimal surfaces satisfying  $K^2 = 2\chi - 2$  and  $p_g \geq 5$ . Note that the case  $p_g = 4$  has been studied in [4]. Let us point out that with our methods we could also recover the classification of [4].

We start with an overview of the case. From the results of [18] the canonical map is not composed with a pencil. Also, by [22], the canonical map has always degree  $\leq 2$ . If the canonical map is birational, then  $p_g \leq 7$ . The bulk of our analysis is the case when the canonical map has degree 2. In this case the canonical image is always a rational surface and we consider the number  $t$  of isolated fixed points of the involution induced by the canonical map. The main results obtained are:

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- *Theorem 3.6:* Let  $S$  be a minimal surface with  $K_S^2 = 2\chi - 2$  and  $p_g \geq 5$ . Then  $S$  satisfies exactly one of the following:
  - (I) the canonical map  $\phi_{K_S}$  is birational and
    - (Ia)  $|K_S|$  is free from base points and  $p_g \leq 7$ , or
    - (Ib)  $|K_S|$  has exactly one (simple) base point and  $p_g = 5$ ;
  - (II) the canonical map factors through an involution  $i$  and the number  $t$  of isolated fixed points of  $i$  is:
    - (IIa)  $t = 0$ , or
    - (IIb)  $t = 2$ , or
    - (IIc)  $t = 4$ .
- Furthermore
  - *Proposition 4.3:* If  $t = 0$ , then  $p_g \leq 12$  and  $S$  is the minimal resolution of a double cover of  $\mathbb{F}_r$ ,  $r \leq 3$ , branched on a curve in  $|8C_0 + 2(5 + 2r)f|$  having  $12 - p_g$  singular points of multiplicity 4 as only essential singularities.
  - *Theorem 5.2:* If  $t = 2$ , then  $p_g \leq 8$  and one of the following occurs:
    - (i)  $S$  is the minimal resolution of a double cover of a *weak Del Pezzo* surface  $T$  of degree  $p_g + 1$  branched on an effective divisor in  $|-4K_T|$  having exactly two (3,3)-points as essential singularities.
    - (ii)  $S$  is the minimal resolution of a double cover of  $\mathbb{F}_r$ ,  $r \leq 2$ , whose branch curve is the union of a curve in  $|8C_0 + (9 + 4r)f|$  with a fibre. The curve has  $8 - p_g$  singular points of multiplicity 4 and another of type (4, 4) and the fibre is tangent to the curve at the (4, 4)-point.
  - If  $t = 4$ , the surface has a unique genus 2 pencil and in *Proposition 6.2* we see the different possibilities for the singularities of the branch locus.

All these types of surfaces, except possibly type *Ib*, do exist. For surfaces of type *I* we refer to Remark 4 and Proposition 3.7. Surfaces of type *IIa*, *IIb* and *IIc* are easily seen to exist using the descriptions as double covers given in Proposition 4.3, Theorem 5.2 and Proposition 6.2.

The paper is organized as follows. In Section 2 some general properties of involutions are recalled. In Section 3 the canonical map of these surfaces is studied, yielding a first division into cases. In the remaining sections each of these cases is described.

*Notation.* We work over the complex numbers. All varieties are projective algebraic. All the notation we use is standard in algebraic geometry. We just recall the definition of the numerical invariants of a smooth surface  $X$ : the self-intersection number  $K_X^2$  of the canonical divisor  $K_X$ , the *geometric genus*  $p_g(X) := h^0(K_X) = h^2(\mathcal{O}_X)$ , the *irregularity*  $q(X) := h^0(\Omega_X^1) = h^1(\mathcal{O}_X)$  and the *holomorphic Euler characteristic*  $\chi(X) := 1 + p_g(X) - q(X)$ .

An *involution* of a surface  $S$  is an automorphism of  $S$  of order 2. We say that a curve singularity is *nonessential* if it is either a double point or a triple point which resolves to at most a double point after one blow-up. Other curve singularities are said to be *essential*. A  $(m, k)$ -*point* of a curve is a point of multiplicity  $m$ , which resolves to an ordinary point of multiplicity  $k$  after one blow-up. We say that a map is *composed with an involution  $i$*  of  $S$  if it factors through the double cover  $S \rightarrow S/i$ .

We do not distinguish between line bundles and divisors on a smooth variety. Linear equivalence is denoted by  $\equiv$  and numerical equivalence by  $\sim$ .

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## 2. INVOLUTIONS ON SURFACES

Let  $S$  be a minimal surface of general type. Given an involution  $i$  on  $S$ , its fixed locus is the union of a smooth curve  $R$  (possibly empty) and of  $t \geq 0$  isolated points  $P_1, \dots, P_t$ . Let  $\pi' : S \rightarrow S/i$  be the quotient map and set  $B'' := \pi'(R)$ . The surface  $S/i$  is normal and  $Q_1 := \pi'(P_1), \dots, Q_t := \pi'(P_t)$  are ordinary double points, which are the only singularities of  $S/i$ . Resolving these singularities we get a commutative diagram

$$(2.1) \quad \begin{array}{ccc} V & \xrightarrow{h} & S \\ \pi \downarrow & & \downarrow \pi' \\ W & \xrightarrow{g} & S/i \end{array}$$

where  $g$  is the minimal desingularization map,  $h$  is the blow up of  $S$  at  $P_1, \dots, P_t$  and  $V$  is obtained by base change and normalization. Notice that the curves  $A_i := g^{-1}(Q_i)$  are  $(-2)$ -curves. Setting  $B := g^*(B'')$ ,  $\pi$  is a double cover whose branch locus  $B'$  is given by:

$$2L \equiv B' := B + \sum_{i=1}^t A_i.$$

We recall the well known formulas (cf. [3], Chapter V, Section 22):

$$(2.2) \quad K_S^2 - t = K_V^2 = 2(K_W + L)^2,$$

$$(2.3) \quad \mathcal{X}(\mathcal{O}_S) = \mathcal{X}(\mathcal{O}_V) = 2\mathcal{X}(\mathcal{O}_W) + \frac{1}{2}L(K_W + L).$$

Since  $\pi^*(2K_W + B) = h^*(2K_S)$  and  $S$  is a minimal surface of general type,  $2K_W + B$  is a nef and big divisor and (see [8], [23], also ([28])):

$$(2.4) \quad (2K_W + B)^2 = 2K_S^2,$$

$$(2.5) \quad K_W(K_W + L) = \frac{1}{2}K_W(2K_W + B) = \frac{1}{2}(K_S^2 - t) - 2\mathcal{X}(\mathcal{O}_S) + 4\mathcal{X}(\mathcal{O}_W),$$

$$(2.6) \quad h^1(2K_W + L) = h^2(2K_W + L) = 0,$$

$$(2.7) \quad t = K_S^2 + 6\mathcal{X}(\mathcal{O}_W) - 2\mathcal{X}(\mathcal{O}_S) - 2h^0(W, \mathcal{O}_W(2K_W + L)),$$

Note that the bicanonical map of  $S$  factors through  $i$  if and only if  $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$  (see e.g. proof of Proposition 2.1 of [23]).

It will be important in what follows to study the divisor  $3K_W + B$ . This divisor is not necessarily nef but, as shown in Proposition 3.9 of [8], it is possible to assume it is nef.

More precisely, from this proposition and its proof we obtain:

**Proposition 2.1** ([8]). *Suppose  $h^0(3K_W + B) \neq 0$ . There exists a birational morphism  $p : W \rightarrow P$  where  $P$  is a smooth surface and an effective divisor  $\bar{B}$  on  $P$  with the following properties:*

- *there are  $t$   $(-2)$ -curves  $C_i$  on  $P$  such that  $p^*(C_i) = A_i$ ,  $i = 1, \dots, t$  and  $\bar{B}$  is disjoint from the union of the curves  $C_i$ ;*
- *there is  $\bar{L}$  in  $\text{Pic}(P)$  such that  $\bar{B} + \sum_1^t C_i = 2\bar{L} = \bar{B}'$  and  $p^*(K_P + \bar{L}) = K_W + L$ ;*
- *the double cover  $\bar{V}$  of  $P$  defined by  $\bar{B} + \sum_1^t C_i = 2\bar{L}$  is a surface with at most Du Val singularities and such that  $V$  is the minimal desingularization of  $\bar{V}$ ;*
- *$p^*(2K_P + \bar{B}) = 2K_W + B$ ;*
- *$3K_P + \bar{B}$  is nef.*

*Proof.* All the above follows easily from the statement and the proof of Proposition 3.9 of [8]. We remark that although Proposition 3.9 of [8] is stated for surfaces  $S$  with  $p_g = 0$ , the proof is valid for any double cover as above.  $\square$

*Remark 1.* It is easily seen that the formulas and properties given above will still hold if we substitute  $W$  with  $P$ ,  $B$  with  $\bar{B}$  and  $L$  with  $\bar{L}$ . Note also that, for  $m \geq 3$ ,  $mK_W + B = p^*(mK_P + \bar{B}) + (m-2)E$ , where  $E$  is the exceptional divisor of  $p$ . Hence  $|mK_W + B| = p^*(|mK_P + \bar{B}|) + (m-2)E$  and thus in particular  $|mK_W + B|$  is composed with a pencil if and only if  $|mK_P + \bar{B}|$  is.

*Remark 2.* In the case where the involution  $i$  has no isolated fixed points and  $H^0(W, 2K_W + L) \neq 0$ , we notice that also  $2K_P + \bar{L}$  will be nef. In fact in that case,  $V = S$  and  $\pi^*(K_W + L) = K_S$ , yielding in particular that  $K_W + L$  is nef and thus also that  $K_P + \bar{L}$  is nef. Suppose, for contradiction, that  $2K_P + \bar{L}$  is not nef. Then there exists an irreducible curve  $E$  such that  $(2K_P + \bar{L})E < 0$ . Since  $2K_P + L$  is a nonzero effective divisor, one must have that  $E^2 < 0$  (see [26], pg 34). On the other hand, since  $K_P + L$  is nef, necessarily  $K_P E < 0$ . So  $E$  is a  $(-1)$ -curve. Then from  $E(2K_P + \bar{L}) < 0$ , we conclude  $E(K_P + \bar{L}) = 0$  and so  $E(3K_P + \bar{B}) < 0$ . This contradicts the construction of  $P$  as above. So  $2K_P + \bar{L}$  is nef.

### 3. THE CANONICAL MAP

In this section  $S$  denotes a minimal surface of general type with  $K_S^2 = 2\chi - 2$  and  $p_g \geq 5$ . First:

**Lemma 3.1.** *Let  $S$  be a minimal surface of general type with  $K_S^2 = 2\chi - 2$  and  $p_g \geq 5$ . Then:*

- $S$  is regular;
- $K_S^2 = 2p_g$ ;
- $S$  has no torsion divisors.

*Proof.* From ([7], Lemma 14), one has that  $q = 0$  and thus  $K_S^2 = 2p_g$ . Also, by ([12], Theorem A), surfaces satisfying  $K_2 = 2\chi - 2$  can only have torsion if  $p_g \leq 4$ .  $\square$

We write  $|K_S| = |H| + F$ , where  $|H|$  is the moving part of  $|K_S|$  and  $F$  the fixed part and let  $\rho : \tilde{S} \rightarrow S$  be a composition of blow-ups such that the variable part  $|\tilde{H}|$  of  $|\rho^*K_S|$  is free from base points. We denote by  $E$  the exceptional divisor of  $\rho$ , by  $F'$  the fixed part of  $|\rho^*K_S|$  and by  $\Sigma$  the canonical image of  $S$ .

**Lemma 3.2.** *Let  $S$  be a minimal surface of general type with  $K_S^2 = 2\chi - 2$  and  $p_g \geq 5$ . Then one of the following occurs:*

- $\phi_{K_S}$  is a birational map;
- $\phi_{K_S}$  is a rational map of degree 2 onto a rational surface.

*Proof.* Note first that  $\phi_{K_S}$  is generically finite because from ([18], Theorem 1.2) surfaces with  $p_g \geq 5$  and  $|K|$  composed with a pencil must satisfy  $K^2 > 4p_g - 7$ . So, the canonical image  $\Sigma$  is an irreducible and nondegenerate surface in  $\mathbb{P}^{p_g-1}$ ,

$$2p_g = K_S^2 \geq (\deg \phi_{K_S})(\deg \Sigma) \geq (\deg \phi_{K_S})(p_g - 2).$$

So  $\deg \phi_{K_S} \leq 3$  and if  $\deg \phi_{K_S} = 3$ , then  $p_g = 5$  or  $p_g = 6$ .

Assume that  $\deg \phi_{K_S} = 3$ . Then a general curve in  $|K_S|$  is smooth, because  $|K_S|$  has no base points if  $p_g = 6$ , or a unique base point if  $p_g =$

5. Since  $\Sigma$  is a surface of minimal degree  $p_g - 2$  in  $\mathbb{P}^{p_g-1}$  we obtain a contradiction to ([22], Theorem 2.1) where it is shown that if the general curve in  $|K_S|$  is smooth and the canonical map is of degree 3 onto a surface of minimal degree  $p_g - 2$  in  $\mathbb{P}^{p_g-1}$ , then  $p_g \leq 5$  and  $K_S^2 \leq 9$ . Therefore, we have  $\deg \phi_{K_S} \leq 2$ .

If  $\deg \phi_{K_S} = 2$ , then  $\Sigma$  is a surface of degree  $\leq p_g$  in  $\mathbb{P}^{p_g-1}$ . From  $p_g \geq 5$ , one obtains  $\deg \Sigma \leq p_g < 2p_g - 4$  and thus, by ([6], Lemme 1.4),  $\Sigma$  has Kodaira dimension  $-\infty$ . Since  $S$  is regular we conclude that  $\Sigma$  is a rational surface.  $\square$

If  $\phi_{K_S}$  has degree 2, then the canonical map factors through an involution  $i$ . In this case, we recall the diagram (2.1)

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ \pi \downarrow & & \downarrow \pi' \\ W & \xrightarrow{g} & S/i \end{array}$$

where  $\pi$  is a double cover with branch locus  $2L \equiv B' = B + \sum_1^t A_i$ . We consider the  $\mathbb{Q}$ -divisor  $B/2$  and we keep the notation of Section 2.

*Remark 3.* Note that if the canonical map factors through an involution, since  $\mathcal{X}(S/i) = 1$  by Lemma 3.2, we have

$$t = 4 - 2h^0(2K_W + L)$$

so the number of isolated fixed points of the involution is  $t = 0, 2$  or  $4$ .

Also, as a consequence of the double cover formulas

**Proposition 3.3.** *Let  $S$  be a minimal surface of general type with  $K_S^2 = 2\chi - 2$  such that the canonical map factors through an involution with rational quotient. Then, with the notation above:*

- (i)  $K_W(K_W + L) = -p_g + 2 - \frac{t}{2}$ ;
- (ii)  $h^0(2K_W + L) = 2 - t/2$ ;
- (iii)  $(2K_W + L)^2 = K_W^2 - p_g - \frac{3}{2}t + 4$ ;
- (iv)  $(2K_W + B)^2 = 4p_g$ ;
- (v)  $(2K_W + B)(2K_W + L) = 4\epsilon + 4 - t$ ;
- (vi)  $K_W L = K_W(B/2) = 2 - p_g - t/2 - K_W^2$ ;
- (vii)  $(B/2)^2 = L^2 + \frac{t}{2} = 3p_g + t + K_W^2 - 4$ ;
- (viii)  $p_a(2K_W + B) = p_g + 3 - \frac{t}{2}$ ;
- (ix)  $h^0(3K_W + B) = p_a(2K_W + B)$ ;
- (x) if  $3K_W + B$  is nef and big then  $h^0(4K_W + B) = p_a(3K_W + B)$ .

*Proof.* All the equalities above, except the two last, are a direct consequence of the formulas in the previous section. In fact, since  $S$  is regular, it satisfies  $\chi(\mathcal{O}_S) = p_g + 1$  and  $W$  being a rational surface satisfies  $\chi(\mathcal{O}_W) = 1$ .

The two last equalities come from the Riemann-Roch theorem since  $2K_W + B$  is nef and big.  $\square$

Let us recall the next result due to Castelnuovo (*cf.* [1]):

**Lemma 3.4** (*Castelnuovo's Bound*). *Let  $C$  be a smooth curve of genus  $g$  that admits a birational mapping onto a nondegenerate curve of degree  $d$  in  $\mathbb{P}^r$ . Let  $m = [(d-1)/(r-1)]$  and  $\varepsilon = (d-1) - m(r-1)$ . Then  $g \leq \pi(d, r)$ , where  $\pi(d, r) = m(m-1)(r-1)/2 + m\varepsilon$ .*

We will need also:

**Lemma 3.5.** *Let  $S$  be a minimal surface of general type such that  $K_S^2 = 3p_g - 5$  and  $q = 0$ . If the canonical map is birational, then  $|K_S|$  does not have a fixed component and has at most one (simple) base point.*

*Proof.* We use the notation introduced in the beginning of this section. Since we are assuming  $\phi_{K_S}$  birational, by [17],  $\tilde{H}^2 \geq 3p_g - 7$ . Thus we have

$$\tilde{H}^2 = 3p_g - 5, \quad 3p_g - 6, \quad \text{or} \quad 3p_g - 7.$$

If  $\tilde{H}^2 = 3p_g - 5$ , the  $|K_S|$  has no base points. If  $\tilde{H}^2 = 3p_g - 6$ , since  $K_S = H + F$  and  $K_S$  is nef and 2-connected, then  $F = 0$  and we have at most one (simple) base point. So, if the statement is not true, then necessarily  $\tilde{H}^2 = 3p_g - 7$  and the general curve  $C$  in  $|\tilde{H}|$  is nonsingular. Note that  $|\rho^*K_S| = |\tilde{H}| + E + F'$ , where  $E$  is the exceptional divisor of  $\rho$  and  $F'$  the fixed part of  $\rho^*K_S$ , then  $\tilde{H}(E + F') > 0$ . So, by the adjunction formula,  $C$  is of genus

$$3p_g - 6 + \frac{1}{2}\tilde{H}(2E + F'),$$

On the other hand, from Lemma 3.4 we obtain  $g(C) \leq 3p_g - 6$ , a contradiction.  $\square$

We can now give a rough classification:

**Theorem 3.6.** *Let  $S$  be a minimal surface with  $K_S^2 = 2\chi - 2$  and  $p_g \geq 5$ . Then  $S$  satisfies exactly one of the following:*

- (I) *the canonical map  $\phi_{K_S}$  is birational and*
  - (Ia)  *$|K_S|$  is free from base points and  $p_g \leq 7$ , or*
  - (Ib)  *$|K_S|$  has exactly one (simple) base point and  $p_g = 5$ ;*
- (II) *the canonical map factors through an involution  $i$  and the number  $t$  of isolated fixed points of  $i$  is:*
  - (IIa)  *$t = 0$ , or*
  - (IIb)  *$t = 2$ , or*
  - (IIc)  *$t = 4$ .*

*Proof.* If  $\deg \phi_{K_S} = 1$ , by the Castelnuovo inequality, one has  $p_g \leq 7$ . Now, using ([21], Lemma 1.3) and ([2], Lemma 1.1) for  $p_g = 6$  and 7 respectively, we obtain that  $|K_S|$  is free from base points. Finally, by Lemma 3.5 for

$p_g = 5$ , the canonical system  $|K_S|$  does not have fixed components and has at most one (simple) base point.

If  $\deg \phi_{K_S} = 2$ , the result follows from Remark 3.  $\square$

*Remark 4.* Surfaces of type  $(Ia)$  with  $p_g = 7$ , have been studied, among others, by Ashikaga and Konno ([2]) and Miranda ([24]). In particular, Miranda has proved that  $\phi_{K_S}$  maps  $S$  into the Veronese cone or into a rational normal scroll.

For surfaces of type  $(Ia)$  with  $p_g = 6$ , we refer to [21]. For this case Konno has shown that the canonical image is contained in a threefold  $W$  of  $\Delta$ -genus  $\leq 1$  which is cut out by all quadrics through the canonical image.

Ciliberto in [11], proves the existence of surfaces of type  $(Ia)$  with  $p_g = 5$ . He has studied the moduli space of such surfaces, its dimension and its unirationality. Furthermore he has shown that the canonical image of a generic such surface has only isolated singularities and cannot lie in a quadric.

Surfaces of type  $(Ib)$ , as far as we know, have not been studied yet and we do not know whether they exist. Next we give some of their properties.

**Proposition 3.7.** *If  $S$  is a surface of type  $(Ib)$ , then:*

- *a general canonical curve  $D$  is smooth and non hyperelliptic of genus 11 ;*
- *the image of  $D$  via the canonical map of  $S$  is a curve  $D_0$  of degree 9 in  $\mathbb{P}^3$  with one double point;*
- *the canonical image of  $S$  is contained in a singular quadric of  $\mathbb{P}^4$ .*

*Proof.* Since by Theorem 3.6  $|K_S|$  has only one simple base point, a general member  $D \in |K_S|$  is irreducible and nonsingular. Thus, from  $K_S^2 = 10$  and the adjunction formula we obtain that the geometrical genus of  $D$ ,  $g(D)$ , is 11. Also  $D$  is nonhyperelliptic because the canonical map of  $S$  is birational. The image  $D_0$  of  $D$  by  $\phi_{K_S}$  is an irreducible nondegenerate curve of degree 9 in  $\mathbb{P}^3$ . Since  $g(D) = 11$ ,  $11 \leq p_a(D_0)$ . On the other hand, applying the main theorem of [10] we obtain  $p_a(D_0) \leq 12$ . If  $p_a(D_0) = 11$ ,  $D_0$  is a nonsingular curve with degree 9 and  $g = 11$  in  $\mathbb{P}^3$ , but this is not possible, (cf. ([14], Exercise 6.4 and Remark 6.4.1). Hence  $p_a(D_0) = 12$  and so  $D_0$  has exactly one singular double point.

For the last assertion consider the subspace  $V$  of  $H^0(2K_S)$  generated by products of sections of  $K_S$ . We claim that  $V$  has dimension 14. By [13, Prop. 3.1]  $\dim V \geq 14$ . Assume for contradiction that  $\dim V \geq 15$ . Since the kernel of the restriction map  $H^0(S, 2K_S) \rightarrow H^0(D, K_D)$  is isomorphic to  $H^0(S, K_S)$  and so 5-dimensional, the image of the restriction of  $V$  to  $D$  is at least 10-dimensional. Let  $x$  be the unique base point of  $|K_S|$ , then every section in  $V$  vanishes at least twice in  $x$ . So we conclude that  $h^0(D, K_D - 2x) \geq 10$  and hence, by the Riemann-Roch theorem and  $g(D) = 11$ ,  $h^0(D, 2x) \geq 2$ , a contradiction because  $D$  is nonhyperelliptic. So



$\dim V = 14$  and since  $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 15$  we conclude that the canonical image  $\Sigma$  of  $S$  is contained in a quadric  $Q$  of  $\mathbb{P}^4$ . Furthermore,  $Q$  must be singular because the degree of  $\Sigma$  is 9 and any surface on a nonsingular quadric of  $\mathbb{P}^4$  is a complete intersection.  $\square$

#### 4. SURFACES OF TYPE (IIa)

We start by stating a general fact:

**Lemma 4.1.** *Let  $D$  be a nef divisor on a rational surface such that  $D^2 = 0$  and  $KD = -2r < 0$ . Then  $|D| = |rN|$ , where  $|N|$  is a base point free pencil of curves of genus 0.*

*Proof.* By the Riemann-Roch theorem,  $h^0(D) \geq r + 1$ .

Write  $|D| = |M| + Z$ , where  $Z$  is the fixed part and  $|M|$  is the moving part. Since  $D$  is nef and  $M$ , being the moving part of  $|D|$  is nef,  $0 = D^2 \geq DM = M^2 + MZ \geq MZ \geq 0$ .

So  $M^2 = MZ = Z^2 = 0$ . In particular  $|M|$  is composed with a pencil, hence,  $M = aN$  with  $h^0(M) = a + 1 \geq r + 1$ . Since  $MZ = 0$ ,  $NZ = 0$  and either  $Z = 0$  or so by Zariski's lemma (see Chp. III, Lemma (8.2) of [3])  $Z = bN$ , with  $b$  a positive rational number. From  $KD = -2r$ , we conclude that  $KN < 0$  and so, by adjunction,  $KN = -2$  and  $KM \geq KD$ . Since  $-2a = KM \leq KD = -2r$ ,  $Z$  must be zero and  $a = r$ , i.e.  $h^0(M) = r + 1$ .  $\square$

Throughout this section, we make the following assumption:

*Assumption 5.*  $S$  is a minimal surface with  $K_S^2 = 2p_g$ ,  $q = 0$ ,  $p_g \geq 5$ , and such that  $\phi_{K_S}$  has degree 2 with  $t = 0$ .

We keep the notation of Section 2, and in particular  $p : S/i \rightarrow P$  is the birational morphism such that  $2K_P + \bar{L}$  is nef (see Remark 2). Then:

**Lemma 4.2.** *Let  $S$  be as in Assumption 5, then the linear system  $|2K_P + \bar{L}|$  is a rational pencil without base points. Moreover,  $|\pi^*(2K_P + \bar{L})|$  is a hyperelliptic pencil of genus 3 in  $S$ .*

*Proof.* Remark first that  $(2K_P + \bar{L})^2 = \alpha \geq 0$ , since  $2K_P + \bar{L}$  is nef. Next, by Proposition 3.3, we get  $(K_P + \bar{L})^2 = p_g$  and  $(K_P + \bar{L})(2K_P + \bar{L}) = 2$ . It follows immediately that  $(K_P + \bar{L} + 2K_P + \bar{L})^2 = p_g + \alpha + 4 > 0$ , so as a consequence of the Index theorem  $p_g \cdot \alpha \leq 4$  and hence  $(2K_P + \bar{L})^2 = 0$ . Also, from Proposition 3.3 we see  $(2K_P + \bar{L})K_P = -2$ . So, as a consequence of Lemma 4.1, we conclude that  $|2K_P + \bar{L}|$  is a base point free pencil of rational curves and, since  $(2K_P + \bar{L})(K_P + \bar{L}) = 2$ , we finish the proof.  $\square$

**Proposition 4.3.** *Let  $S$  be a surface as in Assumption 5. Then  $p_g \leq 12$  and  $S$  is the minimal resolution of a double cover of  $\mathbb{F}_r$ ,  $r \leq 3$ , branched*

on a curve in  $|8C_0 + 2(5 + 2r)f|$  having  $12 - p_g$  singular points (possibly infinitely near) of multiplicity 4 as only essential singularities.

*Proof.* Let  $S \rightarrow P$  be the map of degree 2 with branch curve  $\bar{B} = 2\bar{L}$ , with possibly inessential singularities. Using Lemma 4.2, we know that  $|2K_P + \bar{L}|$  is a genus 0 pencil without base points. Then, we have  $P \neq \mathbb{P}^2$  and, from Proposition 3.3,  $K_P^2 = p_g - 4$ . So contracting  $12 - p_g$   $(-1)$ -curves contained in the fibres of  $|2K_P + \bar{L}|$  we get a birational morphism  $\gamma : P \rightarrow \mathbb{F}_r$ .

Let  $f$  be a fibre of  $\mathbb{F}_r$  and  $C_0$  a section with  $C_0^2 = -r$ . Denote  $\bar{L} = \gamma^*(aC_0 + bf) - \sum c_i E_i$ , since  $2K_P + \bar{L} = \gamma^*(f)$ , then

$$(a - 4)C_0 + (b - 2(2 + r))f + \sum (2 - c_i)E_i = f$$

and we obtain  $a = 4$ ,  $c_i = 2$  and  $b = +5 + 2r$ .

Note that  $c_i = 2$  for every  $i$  means that the essential singularities are quadruple points.

In the end, we can write  $\gamma^*(C_0) = B_0 + \sum \xi_i E_i$ , where  $B_0$  is the strict transform of  $C_0$ . Then  $0 \leq (K_P + \bar{L})B_0 = (2C_0 + (3 + r)f)C_0 - \sum \xi_i \leq 3 - r$  which implies  $r \leq 3$ .  $\square$

*Remark 6.* In Proposition 4.3, if  $p_g \leq 11$  we have also that  $r \leq 2$ . Indeed, if  $r = 3$  the image of  $\bar{B}$  is in  $|8C_0 + 22f|$ . We see that  $C_0$  is in the fixed part of  $|8C_0 + 22f|$  and  $C_0(7C_0 + 22f) = 1$ , so the essential singularities are not contained in  $C_0$ . Since  $p_g \leq 11$  there exists at least one essential singularity on a fibre of the ruling. Blowing up this point and next contracting the strict transform of the fibre, we obtain a new birational morphism from  $P$  onto  $\mathbb{F}_2$  with a quadruple point on the infinity section.

**Corollary 4.4.** *Let  $S$  be as in Proposition 4.3. Assume also that  $p_g \leq 11$  and  $C_0$  is not contained in  $\bar{B}$ . Then there exists a rational map such that the image of  $\bar{B}$  in  $\mathbb{P}^2$  is a curve of degree 14 with  $12 - p_g$  points of multiplicity 4 and one point of multiplicity 6 as unique essential singularities. For  $r = 2$ , the singular point of multiplicity 6 is infinitely near to, at least, a point of multiplicity 4.*

*Proof.* From Proposition 4.3 and Remark 6, up to an elementary transformation of  $F_r$  centered in a quadruple point, we can assume that  $r = 0$  or 2.

If  $r = 0$ , let  $f_1$  and  $f_2$  be the two rulings of  $\mathbb{F}_0$ . Then the image of  $\bar{B}$  in  $\mathbb{F}_0$  is a curve  $\bar{B}_{\mathbb{F}_0} \in |8f_1 + 10f_2|$ . Since  $p_g \leq 11$ , there exists at least one point of multiplicity 4. We blow up one of these points of  $\mathbb{F}_0$  and obtain a new line, a  $(-1)$ -curve; next blow down the two fibres passing through the point. We obtain two singularities of multiplicity 4 and 6; the image of  $\bar{B}_{\mathbb{F}_0}$  meets the new line in  $6 + 4 + 4 = 14$  points. In sum, there exists a birational map  $\mathbb{F}_0 \dashrightarrow \mathbb{P}^2$  such that the image of  $\bar{B}_{\mathbb{F}_0}$  is a curve of degree 14 with two (distinct) points of multiplicity 6 and 4 plus  $11 - p_g$  points of multiplicity 4. Note that some of the essential singular points are possibly infinitely near.

Similarly, if  $r = 2$ , then  $\bar{B}_{\mathbb{F}_2} \in |8C_0 + 18f|$ . Since by assumption  $C_0$  is not contained in  $\bar{B}$ , as before, there exists at least one point of multiplicity 4 not contained in the infinity section, so there exists a birational map  $\mathbb{F}_2 \dashrightarrow \mathbb{P}^2$  such that the image of  $\bar{B}_{\mathbb{F}_2}$  in  $\mathbb{P}^2$  is a curve of degree 14 with a point of multiplicity  $(6, 4)$  plus  $11 - p_g$  points of multiplicity 4. As in the case  $r = 0$  we note that apart from the singular point of multiplicity  $(6, 4)$ , some of the other essential singularities are possibly infinitely near.  $\square$

## 5. SURFACES OF TYPE (IIb)

We recall that  $S$  is a surface of type (IIb) if the canonical map factors through an involution with  $t = 2$ . We can then write the branch curve of the double cover  $V \rightarrow W$  as  $2L \equiv B' = B + A_1 + A_2$ , where  $A_1, A_2$  are  $(-2)$ -curves.

From Proposition 3.3 we have  $h^0(3K_W + B) = p_g + 2$ , so if  $P$  and  $\bar{B}$  are as in Proposition 2.1, the effective divisor  $3K_P + \bar{B}$  is nef.

Also, from Proposition 3.3:

**Lemma 5.1.** *Let  $S$  be a minimal surface with  $K_S^2 = 2p_g$ ,  $q = 0$  and  $p_g \geq 5$ , and  $i$  an involution on  $S$  such that  $t = 2$ . Then  $K_P(\bar{B}/2) = 1 - p_g - K_P^2$  and  $(\bar{B}/2)^2 = \bar{L}^2 + 1 = 3p_g - 2 + K_P^2$ ;*

Throughout this section we will prove the following theorem:

**Theorem 5.2.** *Let  $S$  be a minimal surface with  $K_S^2 = 2p_g$ ,  $q = 0$  and  $p_g \geq 5$ , such that the canonical map factors through an involution with  $t = 2$ . Then  $p_g \leq 8$  and one of the following occurs:*

- (i)  *$S$  is the minimal resolution of a double cover of a weak Del Pezzo surface  $T$  of degree  $p_g + 1$  branched on an effective divisor in  $|-4K_T|$  having exactly two  $(3, 3)$ -points as essential singularities.*
- (ii)  *$S$  is the minimal resolution of a double cover of  $\mathbb{F}_r$ ,  $r \leq 2$ , whose branch curve is the union of a curve in  $|8C_0 + (9 + 4r)f|$  with a fibre. The curve has  $8 - p_g$  singular points (possibly infinitely near) of multiplicity 4 and another of type  $(4, 4)$  and the fibre is tangent to the curve at the  $(4, 4)$ -point.*

*Proof.* We divide the proof into steps.

**Step 1:** *With the usual notation  $K_P^2 = p_g - 2$  or  $K_P^2 = p_g - 3$ .*

From Proposition 3.3  $h^0(2K_W + L) = 1$  and also  $0 \leq (3K_P + \bar{B})(2K_P + \bar{L}) = K_P^2 + 3 - p_g$ , therefore  $K_P^2 \geq p_g - 3$ . On the other hand, by the Index theorem  $K_P^2(\bar{B}/2)^2 \leq (K_P(\bar{B}/2))^2$  and hence, from Lemma 5.1, we get  $K_P^2 \leq \frac{(p_g - 1)^2}{p_g}$  and the assertion follows.

**Step 2:** *If  $4K_P + \bar{B}$  is not nef, there exists a birational morphism  $p_1 : P \rightarrow P_1$  such the divisor  $4K_{P_1} + B_{P_1}$  is nef.*

From Proposition 3.3 and Step 1,  $h^0(4K_P + \bar{B}) > 0$ . So, if  $4K_P + \bar{B}$  is not nef, there is an irreducible curve  $E_1$  such that  $E_1(4K_P + \bar{B}) < 0$  and so as in Remark 2, we can see that  $E_1$  is a  $(-1)$ -curve with  $E_1(3K_P + \bar{B}) = 0$  and so  $E_1\bar{B} = 3$ . Since  $\bar{B}'$  is an even divisor, it is clear that  $E_1(C_1 + C_2) > 0$  and it is an odd number. Since  $(E_1 + C_1 + C_2)(3K_P + \bar{B}) = 0$ ,  $(E_1 + C_1 + C_2)^2 < 0$ , by the Index theorem. As  $(E_1 + C_1 + C_2)^2 = -1 + 2E_1(C_1 + C_2) - 4$  the only possibility is  $E(C_1 + C_2) = 1$ . If, say,  $E_1C_1 = 1$ , when  $E_1$  is contracted the image of  $C_1$  is a  $(-1)$ -curve that is in the branch locus and intersects the image of  $\bar{B}$  at a triple point. So at this point, the image of  $4K_P + \bar{B}$  is not nef any more. Therefore, we have to contract the image of  $C_1$  also, so the inductive step consists in contracting twice. If necessary, we can repeat the same argument for another  $(-1)$ -curve  $E_2$  with  $E_2C_2 = 1$ , obtaining the result.

To continue with the proof, we analyse the two values of  $K_P^2$ .

**Step 3:** *If  $K_P^2 = p_g - 3$ , then  $S$  is the minimal resolution of a double cover of a weak Del Pezzo surface  $T$  of degree  $p_g + 1$  branched on a divisor in  $|-4K_T|$  having two  $(3,3)$ -points.*

Let  $p_1 : P \rightarrow P_1$  be the birational morphism such that  $4K_{P_1} + B_{P_1}$  is nef. If  $s$  is the number of  $(-1)$ -curves contracted by  $p_1$ , from Lemma 5.1,  $(4K_P + \bar{B})^2 = -4$  and one has  $s \geq 4$ . Besides, note that  $K_{P_1}^2 = K_P^2 + s$  and  $(B_{P_1}/2)^2 = (\bar{B}/2)^2 + \frac{9}{4}s$  by Lemma 5.1, so

$$(5.1) \quad 0 \leq (2K_{P_1} + B_{P_1})(4K_{P_1} + B_{P_1}) = 4 - s$$

hence,  $s = 4$ ; since  $4K_{P_1} + B_{P_1}$  is an effective divisor and by the Index theorem we have that  $2K_{P_1} + B_{P_1}/2$  is a trivial divisor. Therefore  $-K_{P_1} = K_{P_1} + B_{P_1}/2$  gives  $-K_{P_1}$  nef and big, so  $P_1$  is a weak Del Pezzo surface of degree  $K_{P_1}^2 = p_g + 1$ . Finally, let us analyse the image of  $C_1$  and  $C_2$  in  $P_1$ . As we have seen,  $C_1$  and  $C_2$  are  $(-2)$ -curves in  $P$ , however, if  $E_1$  is a  $(-1)$ -curve contracted by  $p_1$ , by Step 2, we can suppose that  $E_1C_1 = 1$  and  $C_1$  becomes a  $(-1)$ -curve, whose intersection with the image of  $\bar{B}$  is equal to 3 and it will be contracted as well. Since  $s = 4$ , we can repeat the same argument for another  $(-1)$ -curve  $E_2$  with  $E_2C_2 = 1$ , obtaining the two singular  $(3,3)$ -points.

*Remark 7.* Notice that Theorem 4.2 of [4] gives the same result as in Step 3 for the case  $p_g = 4$ .

**Step 4:** *If  $K_P^2 = p_g - 2$ , then  $|4K_P + \bar{B}|$  is a rational pencil without base points.*

Keeping the same notation as in the proof of Step 3, and using the Index theorem we obtain that  $K_{P_1}^2(B_{P_1}/2)^2 \leq (K_{P_1}(B_{P_1}/2))^2$ , which implies  $s \leq$

$\frac{4}{p_g+2}$  by Lemma 5.1; hence  $s = 0$  and we conclude that  $4K_P + \bar{B}$  is nef. From Proposition 3.3, we see that  $(4K_P + \bar{B})^2 = 0$ ; besides, from Lemma 5.1, we have  $K_P(4K_P + \bar{B}) = -2$ . So, applying Lemma 4.1, the result follows.

**Step 5:** *If  $K_P^2 = p_g - 2$ , then  $S$  is the minimal resolution of a double cover of  $\mathbb{F}_r$ ,  $r \leq 2$ , whose branch curve is the union of a curve in  $|8C_0 + (9+4r)f|$  with a fibre. The curve has  $8-p_g$  singular points of multiplicity 4 and another of type  $(4, 4)$  and the fibre is tangent to the curve at the  $(4, 4)$ -point.*

From Step 4,  $|4K_P + \bar{B}|$  is a genus 0 pencil without base points, hence  $P \neq \mathbb{P}^2$  and contracting  $10 - p_g$  exceptional curves, we get a birational morphism  $\gamma : P \rightarrow \mathbb{F}_r$ .

Let  $f$  be a fibre of  $\mathbb{F}_r$  and  $C_0$  a section with  $C_0^2 = -r$ . Write  $\bar{B} = \gamma^*(aC_0 + bf) - \sum c_i E_i$ , since

$$4K_P + \bar{B} = \gamma^*((a-8)C_0 + [b-4(2+r)]f) + \sum (4-c_i)E_i = \gamma^*(f)$$

we obtain  $\bar{B} = \gamma^*(8C_0 + (9+4r)f) - 4\sum E_i$ .

Also, from  $0 \leq \gamma^*(C_0)(2K_P + \bar{B})$ , we have  $r \leq 2$ .

By Proposition 2.1, we know that  $C_1$  and  $C_2$  are  $(-2)$ -curves on  $P$  and hence  $C_i(4K_P + \bar{B}) = 0$ , so they are contained in the fibres. More precisely, since  $C_i(2K_P + \bar{L}) = -1$ , we can write  $2K_P + \bar{L} = D + C_1 + C_2$  where  $D$  is an effective divisor with  $h^0(D) = 1$ . Hence,  $4K_P + \bar{B} = 2D + C_1 + C_2$ , so  $C_1$  and  $C_2$  are in the same fibre. By easy calculations we have  $D^2 = -1$  and  $K_P D = -1$ . Then it is easy to see that contracting  $D$  and then, say, the image of  $C_1$ , we obtain a singularity of multiplicity  $(4, 4)$  of the image of  $\bar{B}$ , such that the fibre passing through this point is contained in the branch locus and it is tangent to  $\bar{B}$  at the  $(4, 4)$ -point. Finally, since there are  $10 - p_g$  singular points of multiplicity 4, then  $p_g \leq 8$ .  $\square$

To end this section we make several remarks.

*Remark 8.* If  $S$  is a surface as in Step 5, we can proceed as in the proof of Corollary 4.4. First, by Step 5 we have  $r \leq 2$ , so we can suppose that  $r = 0, 2$ . If  $r = 0$ , there exists a birational map  $\mathbb{F}_0 \dashrightarrow \mathbb{P}^2$  such that we can see the image of the branch  $B'$  on  $\mathbb{P}^2$  as a curve of degree 14 of type  $C + l$ , where  $l$  is a line and  $C$  is a curve of degree 13 having two (distinct) points  $P_1$  and  $P_2$  of multiplicity 5 and  $(4, 4)$  respectively at the intersection with  $l$ , plus  $8 - p_g$  points of multiplicity 4, and no further essential singularities. For the case  $r = 2$  it is easily seen that the infinity section  $C_0$  cannot contain the  $(4, 4)$ -point. Then with the same procedure the case  $r = 2$  can be expressed as a degeneration of the case  $r = 0$ . The degeneration consists of having  $P_1$  infinitely near to  $P_2$ . Note also that  $l$  is tangent to  $C$  at  $P_2$ .

## 6. SURFACES OF TYPE $(IIc)$

Finally, we are going to study surfaces with  $K_S^2 = 2p_g$ ,  $q = 0$  and  $p_g \geq 5$ , such that the canonical map factors through an involution with  $t = 4$ .

For these surfaces we have that  $h^0(2K_W + L) = 0$ . Thus the bicanonical map of  $S$  is composed with  $i$ , hence not birational. Since by hypothesis  $K_S^2 \geq 10$ , using ([27], Proposition 3),  $S$  has a pencil of curves of genus 2, necessarily rational because  $q = 0$ . We remark that the existence of the genus 2 pencil can be also checked directly by considering the linear system  $|3K_W + B|$ .

*Remark 9.* It is easy to see that the rational pencil of curves of genus 2 is unique. Otherwise, let  $|G_1|$  and  $|G_2|$  be two pencils of genus 2 without base points; since  $G_1G_2 \geq 2$ ,  $(G_1 + G_2)^2 \geq 4$ . Since  $K_S(G_1 + G_2) = 4$ , we obtain  $K_S^2(G_1 + G_2)^2 - (K_S(G_1 + G_2))^2 > 0$ , a contradiction to the Index theorem, because  $K_S^2 \geq 10$ .

*Remark 10.* Recall that if a surface has a pencil of genus 2, there exists a map of degree 2 onto a ruled surface, mapping each genus 2 fibre by its canonical map onto a fibre of the ruling. Horikawa in [15] proved that with elementary transformations it is possible to obtain a minimal model whose branch locus has only singularities of the following types:  $(0)$ ,  $(I_k)$ ,  $(II_k)$ ,  $(III_k)$ ,  $(IV_k)$  and  $(V)$  (in Horikawa's notation).

In what follows, we will say that a rational fibre of Horikawa's model is *a singular fibre of type*  $(I_k)$ ,  $(II_k)$ ,  $(III_k)$ ,  $(IV_k)$  or  $(V)$  if the branch locus has the corresponding singularity or singularities on the fibre. We will also say that a genus 2 fibre is *of type*  $(I_k)$ ,  $(II_k)$ ,  $(III_k)$ ,  $(IV_k)$  or  $(V)$  if its corresponding rational fibre is of this type.

The next result is well known but for completeness we include its proof.

**Lemma 6.1.** *Let  $S$  be a minimal algebraic surface of general type with  $p_g \geq 4$ ,  $q = 0$  and canonical map not composed with a pencil. If  $S$  has a pencil of genus 2, then the canonical map has degree 2.*

*Proof.* Let  $|G|$  be the genus 2 pencil on  $S$  and note that  $K_S|_G \simeq \omega_G$ . Since  $\omega_G$  is a  $g_2^1$  on  $G$ ,  $\phi_{K_S}$  has even degree.

Since  $h^0(S, K_S) \geq 4$  and  $h^0(G, \omega_G) = 2$ , from the long exact sequence:

$$0 \rightarrow \mathcal{O}_S(K_S - G) \rightarrow \mathcal{O}_S(K_S) \rightarrow \mathcal{O}_S(K_S)|_G \rightarrow 0,$$

we have that  $h^0(S, K_S - G) \geq 2$ . Therefore,  $\phi_{K_S}$  separates the fibres and we obtain the result.  $\square$

**Proposition 6.2.** *The pencil of genus 2 in  $S$  is the pull-back of a ruling of the canonical image  $\Sigma$  of  $S$ . Moreover, the essential singularities of the branch locus in Horikawa's model are of type:  $(I_k)$ ,  $(II_k)$ ,  $(III_k)$ ,  $(IV_k)$ , with  $k = 1, 2$ , and  $(V)$ .*

**Proof:** Since  $K_S|_G = \omega_G$  and  $|K_S|$  is not composed with a pencil, the image of a general element of  $|G|$  is a line, and so a ruling of  $\Sigma$ . Using ([15],

Theorem 3) we get

$$(6.1) \quad 4 = \sum_k \{(2k-1)(\nu(I_k) + \nu(III_k)) + 2k(\nu(II_k) + \nu(IV_k))\} + \nu(V),$$

where  $\nu(*)$  denotes the number of singularities of type  $(*)$ . So we immediately obtain that  $k = 1, 2$ .  $\square$

*Remark 11.* Looking carefully at the resolution of the singular fibres as in Proposition 6.2, we obtain that:

- (i) each singular fibre of type  $(I_1)$  or  $(III_1)$  or  $(V)$  corresponds to one base point of  $|K_S|$  and one isolated fixed point of the involution;
- (ii) each singular fibre of type  $(I_2)$  or  $(III_2)$  corresponds to a fixed component plus one base point of  $|K_S|$ , and three isolated fixed points of the involution;
- (iii) each singular fibre of type  $(II_1)$  or  $(IV_1)$  corresponds to a fixed component of  $|K_S|$ , and there are two isolated fixed points of the involution;
- (iv) finally, each singular fibre of type  $(II_2)$  or  $(IV_2)$  corresponds to a fixed component plus two base points of  $|K_S|$ , and four isolated fixed points of the involution.

We point out that all the fixed components of  $|K_S|$  are  $(-2)$ -curves.

Using double or bidouble covers, it is not difficult to find examples, for instance,

**Example 6.1.** In  $\mathbb{F}_0$ , let  $f_1$  and  $f_2$  denote general fibres of each ruling. Consider the bidouble cover  $\pi : S \rightarrow \mathbb{F}_0$  with smooth branch curves  $D_1 \in |f_1 + f_2|$ ,  $D_2 \in |f_1 + 3f_2|$  and  $D_3 \in |3f_1 + (2a+1)f_2|$ ,  $a \geq 2$ . Using the bidouble cover formulas (see [9], [25]), the surface  $S$  has the invariants  $K^2 = 4a + 2$ ,  $p_g = 2a + 1$  and  $q = 0$ . Now, we analyse the double cover as a composition of two double covers. First, the double cover with branch curve  $D_1 + D_2$ , this is  $Y \rightarrow \mathbb{F}_0$ , where  $Y$  is a rational surface with four  $(-2)$ -curves coming from the intersection points of  $D_1$  and  $D_2$ . The linear system  $|\pi^*f_2|$  is the pencil of genus 2, whose fibres in the general case will be of type  $(I_1)$ . It is easy to see that a mild degeneration of the construction will yield fibres of type  $(III_1)$  or  $V$ .

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